Theory and Gyro-fluid Simulations of Edge-Localized-Modes

X.Q. Xu\(^1\), P.W. Xi\(^1,2\), A. Dimits\(^1\), I. Joseph\(^1\), M.V. Umansky\(^1\), T.Y. Xia\(^1,3\), B. Gui\(^1,3\), S.S. Kim\(^4\), G.Y. Park\(^4\), T. Rhee\(^4\), H. Jhang\(^4\), P.H. Diamond\(^4,5\), B. Dudson\(^6\), P.B. Snyder\(^7\) (\(^1\)LLNL, \(^2\)PKU, \(^3\)ASIPP, \(^4\)NFRI, \(^5\)UCSD, \(^6\)U.York, \(^7\)GA)

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Principal Results

- First order FLR corrections from "gyro-viscous cancellation" in two-fluid model are necessary to agree with gyro-fluid results for high ion temperature.

- Higher ion temperature introduces more FLR stabilizing effects, thus reduces ELM size.

- Developed a fast non-Fourier method for the computation of Landau-fluid closure terms;
  - Implemented the fast non-Fourier method through the solution of matrix equations in which the matrices are tridiagonal or narrowly banded;

- Implemented 3+0 \{n, u_\parallel, P_\parallel\} & 3+1 \{n, u_\parallel, P_\parallel, P_\perp\} electrostatic model equations;
- Implemented 1+0 (n) & 2+0 \{n, u_\parallel\} electromagnetic model for ELM simulations;

- Benchmarked linear GLF simulations with eigen-value calculations
- Benchmarked with other two-fluid codes;
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- Edge 3-field gyro-fluid models for type-I ELMs
  - In long-wavelength limit, this set of gyro-fluid equations is reduced to previous 3-field two-fluid model with additional gyro-viscous terms resulting from the incomplete “gyro-viscous cancellation” in two-fluid model given by Xu et al [2].
  - Utilizing the Padé approximation for the modified Bessel functions, this set of gyro-fluid equations is implemented in the BOUT++ framework with full ion FLR effects.
  - An assumption of an ion steady-state with subsonic flow velocity leads to a model that the ion response is adiabatic for both equilibrium and axisymmetric component of fluctuations, such as $Z_i e\langle \Phi \rangle = T_i \ln \langle P_i \rangle$.

- Edge 4-field gyro-fluid models for small ELMs
- A fast non-Fourier method for the computation of Landau-fluid closure terms
- BOUT++ core gyrofluid simulations of ion temperature gradient turbulence

3-field isothermal gyrofluid model* for ELM simulation: consider the large density gradient at H-mode pedestal

\[
\frac{d \sigma}{dt} = B \nabla_J + 2b_0 \times \kappa \cdot \nabla \tilde{P} + \mu_{i||} \partial_{\parallel}^2 \sigma
\]

\[
\frac{d \tilde{P}}{dt} + V_E \cdot \nabla P_0 = 0
\]

\[
\frac{\partial A}{\partial t} + \partial_{||} \phi_T = \frac{\eta}{\mu_0} \nabla^2 A_{||} - \frac{\eta_H}{\mu_0} \nabla^4 A_{||}
\]

\[
\sigma = \frac{m_i n_0}{B} \left( \nabla^2 \phi + \frac{1}{en_0} \nabla^2 \tilde{P} + \frac{1}{n_0} \nabla n_0 \cdot \nabla \phi \right)
\]

Padé approximation

\[
\begin{align*}
\Gamma_0^{1/2} & \approx \frac{1}{1 + b/2} \\
\Gamma_0 & \approx \frac{1}{1 + b}, \quad b = k_{\perp} \rho_i \\
\Gamma_0 - \Gamma_1 & \approx 1
\end{align*}
\]

Twofluid equations

Relation between twofluid vorticity and gyrokinetic vorticity

\[
\bar{\omega} \approx \sigma_G + \frac{1}{2en_0} \nabla^2 \tilde{P}_{iG}
\]

*) P. B. Snyder and G. W. Hammett, Phys. Plasmas 8, 3199 (2001)
In the presence of large density gradient, gyro-fluid and two-fluid model show qualitative difference when $k_{\perp} \rho_i$ is large.

Consider the large density gradient at H-mode pedestal:

- Two-fluid model: **no stabilizing** on high-$n$ modes,
- Gyro-fluid model: **strong FLR stabilizing** on high-$n$ modes.

What causes the disappear of stabilizing in twofluid model?
Ion-Density-Gradient mode in twofluid model

→ The instability does not localize at peak pressure gradient region
→ Not pressure gradient driven ballooning mode, but other instability
→ Lowest order ballooning equation changes

\[
\frac{1}{J} \frac{\partial}{\partial \chi} \left[ \frac{k_{\perp}^2}{B^2 J} \frac{\partial}{\partial \phi} \right] - \frac{\omega_J}{\rho_i^2 V_A} \left[ \frac{i}{B J} \frac{\partial}{\partial \chi} \hat{\phi} \right] = - \left[ \frac{k_{\perp}^2 \omega}{V_A^2} \left( \omega + \omega_{*i} + \frac{\omega}{k_{\perp}^2 n_0} \frac{dn_0}{d\psi} \nabla q \right) + 2 \frac{\omega_{\perp} \omega_{*i}}{V_A^2 \rho_i^2} \right] \hat{\phi}
\]

Twofluid local dispersion relation

ion diamagnetic stabilizing on ballooning modes

\[
\gamma = \frac{1}{\sqrt{C}} \left( \sqrt{\gamma_{\perp}^2 - \frac{\omega_{*i}^2}{4} \cos \frac{\alpha}{2} + \frac{\omega_{*i}}{2} \sin \frac{\alpha}{2}} \right)
\]
\[
C^2 = 1 + \frac{k_{\perp}^2}{k_{\perp}^2 L_n^2}, \quad \cos \alpha = \frac{1}{C}, \quad \sin \alpha = \frac{1}{C} \frac{k_{\perp}^2}{k_{\perp}^2 L_n}
\]

When \( \omega_{*i} \) increases:

- Ion diamagnetic effect stabilizes ballooning modes \( \Rightarrow \) first term decreases
- Ion density gradient introduces **Ion-Density-Gradient mode**: second term become dominant
Gyroviscous terms are necessary to stabilize Ion-Density-Gradient modes and should be kept in two-fluid model.

Only ion diamagnetic effect in two-fluid model is not sufficient to represent FLR stabilizing if density gradient is large!

- At long wavelength limit, gyro-fluid goes back to two-fluid but with additional gyroviscous terms

\[
\frac{d \omega_G}{dt} + \nabla \cdot \nabla \omega_{G0} - eB(V_{\phi T} - V_{ET}) \cdot \nabla n_{iG} = \frac{d \omega}{dt} + \frac{1}{2\omega_{ci}} \left\{ \nabla^2 [\phi, P_i] - [\nabla^2 \phi, P_i] - [\phi, \nabla^2 P_i] \right\}
\]

- Gyroviscous terms [1] represent necessary FLR effect to stabilize IDG modes and should be kept in two-fluid model
- If without gyroviscous terms, IDG mode will lead to much larger ELM crash in nonlinear phase

Without gyroviscous terms, IDG mode leads to larger ELM crash and more energy loss at nonlinear phase.

Pressure perturbation at ELM crash time:

- With gyroviscous terms:
  - T0 = 4 keV
  - ELM size: 0.06

- W/O gyroviscous terms:
  - T0 = 4 keV
  - ELM size: 0.13
In isothermal limit, linear relation for n=0 component of electric field cannot get nonlinear saturation

\[ \langle \vec{\omega}_T \rangle = 0 \]

Net zonal flow is set to be zero:

\[ \langle \vec{\omega}_0 \rangle = 0 \Rightarrow \phi_0 = -\frac{T_0}{e} \ln P_{i0} \]

Equilibrium part:

Linear relation:

\[ \langle \vec{\omega} \rangle = \frac{n_{i0} m_i}{B} \left( \nabla_{\perp}^2 \phi_{dc} + \frac{1}{n_{i0}} \nabla n_{i0} \cdot \nabla \phi_{dc} + \frac{1}{n_{i0} e} \nabla_{\perp}^2 \tilde{P}_{i,dc} \right) = 0 \Rightarrow \text{linear} \quad \phi_{dc} \]

Net zonal flow is set to be zero:

Strong n=0 EXB:
- Smooth perturbation in poloidal direction
- Reduce radial transport

Very weak n=0 EXB:
- Keep streamer like structure
- Cannot reduce radial transport
- No saturation

\[ \checkmark \text{n=0 component of electric field determines the saturation phase;} \]
\[ \checkmark \text{This linear relation is not correct.} \]
Nonlinear relation generates larger EXB shearing at pedestal top to get the saturation phase.

\[
\langle \sigma_T \rangle = \frac{n_i m_i}{B} \left( \nabla^2 (\phi_{dc} + \phi_0) + \frac{1}{n_i} \nabla n_i \cdot \nabla (\phi_{dc} + \phi_0) + \frac{1}{n_i e} \nabla^2 (\tilde{P}_{i,dc} + P_{i0}) \right) = 0
\]

\[
\Rightarrow \phi_{dc} = -\frac{T_0}{e} \ln \left( \frac{\tilde{P}_{i,dc} + P_{i0}}{P_{i0}} \right) \quad \text{(nonlinear relation)}
\]

Pedestal bottom: n=0 EXB is still strong due to 1/n0

Pedestal top: n=0 EXB flow is enough to reduce radial transport and generate saturation.
Higher ion temperature introduces more FLR stabilizing effects, thus reduces ELM size

- Hyper-resistivity is necessary to ELM crash, but ELM size is weakly sensitive to hyper-resistivity;
- With fixed pressure profile, high ion temperature introduce stronger FLR effect and thus leads to smaller ELM size

![Graph showing ELM size vs. ion temperature](image)

\[ \Delta_{ELM}^{h} = \frac{\Delta W_{ped}}{W_{ped}} = \frac{\langle \int_{R_{in}}^{R_{out}} \int dR \theta (P_{0} - \langle P \rangle_{\xi}) \rangle_{t}}{\int_{R_{in}}^{R_{out}} \int dR \theta P_{0}} \]

(Without density gradient in vorticity)
Equilibrium EXB shear flow can stabilize high-n ballooning modes and reduce ELM size, but introduces Kelvin-Helmholtz instability and leads to larger ELM when flow shear is too large.

\[
\frac{\partial \omega}{\partial t} + \mathbf{V}_{\times B} \cdot \nabla \omega + \mathbf{V}_1 \cdot \nabla \omega_0 = B_0 \nabla \parallel J_\parallel + 2b_0 \times \kappa \cdot \nabla P
\]

\[
\mathbf{V}_{\times B} = \frac{b \times \nabla \Phi_{dia0}}{B} + \frac{b \times \nabla \Phi_{V0}}{B} + \frac{b \times \nabla \phi}{B}
\]

\[
\Phi_{dia0} = -\frac{P_{\text{th}}}{Zcn_{i0}}, \quad \frac{d\Phi_{V0}(\psi)}{d\psi} = D_0[1 - \tanh(D_s(x - x_0))] + C
\]

2+0 isothermal gyrofluid model for ELM simulation

\[ \frac{d\omega_G}{dt} - eB(\mathbf{V}_\Phi - \mathbf{V}_E) \cdot \nabla n_{iG} = B\nabla || J || + 2b_0 \times \kappa \cdot \nabla \tilde{P} - B\nabla || en_0(\tilde{u}_{||e} - \tilde{u}_{||i}) + eB(\nabla || - \nabla ||)n_0\tilde{u}_{||i} \]
\[ \frac{dP_G}{dt} + T_i(\mathbf{V}_\Phi - \mathbf{V}_E) \cdot \nabla n_{iG} + \nabla || P_0\tilde{u}_{||i} - \frac{T_0}{e}\nabla || J || + \nabla || \frac{P_0}{2}(\tilde{u}_{||e} - \tilde{u}_{||i}) + (\nabla || - \nabla ||)\frac{P_0\tilde{u}_{||i}}{2} = 0 \]
\[ m_i n_0 \frac{d\tilde{u}_{||i}}{dt} + m_i n_0(\mathbf{V}_\Phi - \mathbf{V}_E) \cdot \nabla \tilde{u}_{||i} + \partial || P_G + en_0 \frac{\partial (\tilde{A} || - A ||)}{\partial t} + (\partial || - \partial ||)\tilde{p}_i + (\delta \tilde{b} - \delta \tilde{b}) \cdot \nabla \frac{P_0}{2} = 0 \]
\[ \frac{\partial A ||}{\partial t} + \partial || \phi - \frac{1}{n_0e} \partial || P_e = \eta J || + \eta_H \nabla^2 \tilde{J} \]
\[ \omega_G = eB(\Gamma_0^{1/2}\tilde{n}_{iG} - n_0(1 - \Gamma_0)\frac{e\phi}{T_0} + \frac{e\rho_i^2}{T_0} \nabla n_0 \cdot \nabla ((\Gamma_0 - \Gamma_1)\phi) - \tilde{n}_{iG}) \]

where \( \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_E \cdot \nabla \), \( \mathbf{V}_E = \frac{1}{B}b_0 \times \nabla \phi \), \( \mathbf{V}_\Phi = \frac{1}{B}b \times \nabla \Phi = \frac{1}{B}b \times \nabla \Gamma_0^{1/2} \phi \), \( J || = J_0 + \tilde{J} || \),
\( J_0 || = -en_0e u_0 || e \), \( \tilde{u}_i = \Gamma_0^{1/2}\tilde{u}_{||i} \), \( \delta \tilde{b} = \frac{1}{B}\nabla A || \times b \), \( \delta \tilde{b} = \frac{1}{B}\nabla \tilde{A} || \times b \), \( \tilde{A} || = \Gamma_0^{1/2} A || \), \( \nabla || f = B\partial || f \frac{B}{B}, \partial || = \)

- Including ion acoustic wave
  → drift ballooning modes*
- May appear at high-n region → gyrofluid

Drift ballooning mode: Instability for small ELM? Probably not!

- Ion parallel motion → ion acoustic wave
- Electron pressure in Ohm’s law → electron drift wave

Local dispersion relation:

\[ \omega_{*i} = -\omega_{*e} = \frac{1}{2B n_0 e} b \times \nabla P_0 \cdot \mathbf{k}_\perp \]

\[ \omega_s = c_s / R_q \]

Wave resonance condition

\[ f_{res} = |(1 + 2q^2)\omega_s^2 - \omega_{*i}(\omega_{*i} - \omega_{*e})| \]

\[ f_{res} = 0 \rightarrow \text{wave resonance} \]

Implied from local theory

Schematic view of kinetic effects on IBM growth rate

NO drift ballooning mode!

Conditions for drift ballooning mode are difficult to satisfy in real discharges

Local dispersion relation

\[ -\omega (\omega - \omega_{*i}) [(1 + 2q^2)\omega_s^2 - \omega (\omega - \omega_{*e})] = \gamma_I^2 [\omega_s^2 - \omega (\omega - \omega_{*e})] \]

\[ \delta \omega = \omega - \omega_{*i} \]

\[ \Delta \omega^2 = (1 + 2q^2)\omega_s^2 - \omega_{*i}(\omega_{*i} - \omega_{*e}) \]

\[ \gamma \propto \sqrt{(\omega_{*i}\Delta \omega^2)^2 - 8q^2(1 + 2q^2)\omega_{*i}(2\omega_{*i} - \omega_{*e})\omega_s^2\gamma_I^2} \]

\[ \omega_{*i} \sim \omega_{*e} \sim \omega_s \]

\[ \frac{\delta \omega}{\omega_{*i}} \sim \frac{|\Delta \omega^2|}{\omega_{*i}^2} \sim \frac{\gamma_I}{\omega_{*i}} \sim \epsilon \ll 1 \]

• Conditions for drift ballooning
  
  A: Finite local ideal MHD growth rate
  B: Wave resonant condition is satisfied at the finite pressure gradient region

  n=10  \rightarrow  B \text{ not } A
  n=50  \rightarrow  A \text{ not } B

In real discharges, these two conditions are difficult to satisfy simultaneously.
Ion acoustic wave and electron drift wave have stabilizing effect on P-B mode.

Drift ballooning mode is unlikely the instability triggering small ELM in real discharges (DIII-D, C-Mod, NXTX, ITER).
Two-fluid 4-field HHM electromagnetic model
Finite Lamor Radius effects stabilize p-b modes.

\[ \frac{\partial P}{\partial t} = -V_{E} \cdot \nabla P + \frac{\beta}{1 + \beta / 2B_{0}^{2}} \left( \frac{1}{B_{0}} \nabla \left( \frac{B_{0}^{2}}{\beta} \right) \right) - 2b_{b} \times b_{b} \left( \nabla \phi - \frac{\delta_{b}}{B_{0}} \nabla P \right), \]

\[ \frac{\partial V_{1}}{\partial t} = -V_{E} \cdot \nabla V_{1} - \frac{1}{2} \nabla \cdot P, \]

\[ \frac{\partial U}{\partial t} = -V_{E} \cdot \nabla U - B_{0}^{2} \nabla \cdot J + b_{b} \times b_{b} \nabla P + \frac{\delta_{b}}{2B_{0}^{2}} \left[ V_{E} \cdot \nabla (V_{E}^{2} P) - V_{E}^{2} (V_{E} \cdot \nabla P) - V_{E} \cdot \nabla (V_{E}^{2} \phi) \right], \]

\[ \frac{\partial \psi}{\partial t} = -\frac{V_{E} \cdot (B_{0} \phi)}{B_{0}} + \frac{\delta_{b}}{B_{0}^{2}} \nabla \cdot P + \frac{1}{S} \nabla \cdot \nabla \psi + \frac{1}{S_{H}} \nabla \cdot \nabla \psi, \quad S = \mu_{0} L V_{E} / \eta, \quad S_{H} = \mu_{0} L V_{A} / \eta_{H}, \]

where \( B_{0} \) is normalized variable

\[ \delta = \frac{1}{2 \Omega_{e}^{2}}, \quad \Omega_{e} = \frac{eB}{m_{e}}, \quad \tau = T_{e} / T_{e}, \quad \delta_{i} = \frac{\tau}{1 + \tau}, \quad \delta_{e} = \frac{1}{1 + \tau}, \quad \beta = \frac{2 \mu_{0} P_{e}}{B_{0}^{2}}, \]

\[ J = \nabla \cdot \psi, \quad A_{i} = B_{0} \psi, \quad \Phi = B_{0} \phi, \quad d / dt = \partial / \partial t + V_{E} \cdot \nabla, \quad \nabla_{i} = \delta_{b} - b \times \nabla \psi \cdot \nabla, \]

\[ V_{E} = b_{b} \times \nabla \phi, \quad V_{D} = b_{b} \times \nabla P, \quad U = \nabla \cdot \phi + \frac{\partial}{\partial b_{b}^{2}} \nabla \cdot T_{E}^{2} P. \]
Accurate non-Fourier methods for Landau-fluid operators

Tokamak edge:

- kinetic effects important -> need Landau-fluid (LF) operators

\[ \gamma \propto -v_{\text{char}} |k| \]

- Large spatial inhomogeneities & complicated boundary
  - need non-Fourier implementation
  - Useful accurate approximation:

\[ \frac{1}{|k|} \approx \sum_{n=0}^{N} \frac{\alpha^n k_0}{k^2 + \left(\alpha^n k_0\right)^2} \]

- The new method has Fourier-like computational scaling

\[ \checkmark \text{The error is less than } 1.5\%. \]
• For small number of grid cells, direct matrix multiplication is as efficient as Fourier
• Non-Fourier, with fixed N, scales as $N_z$, c.f. $N_z^2$ for direct convolution
• Crossover point is at $N_z \approx 100 - 200 \Rightarrow$ advantage for $N_z \geq 100 - 200$. 
The linear response function matches the published results from HP90 paper, hence the code and scheme must be correct.

\[
\frac{\partial}{\partial t} n + \frac{\partial}{\partial z} (un) = 0
\]

\[
\frac{\partial}{\partial t} (mnun) + \frac{\partial}{\partial z} (umnun) = -\frac{\partial}{\partial z} p + enE - \frac{\partial}{\partial z} S
\]

\[
\frac{\partial}{\partial t} p + \frac{\partial}{\partial z} (up) = -(\Gamma - 1)(p + S) \frac{\partial}{\partial z} u - \frac{\partial}{\partial z} q
\]

\[
q_k = -n_0 \chi_1 \frac{\sqrt{2V}}{|k|} ikT_k
\]

\[
S_k = -mn_0 \mu_1 \frac{\sqrt{2V}}{|k|} iku_k
\]

\[
\chi_1 = \frac{2}{\sqrt{\pi}}
\]

\[
\mu_1 = 0
\]

\[
\Gamma = 3
\]

\[
\frac{1}{|k|} \approx \beta \sum_{n=0}^{N} \frac{\alpha^n}{k^2 + \alpha^{2n}}
\]

\[
q_k = \sum_{n=0}^{N} q^n_k = -\chi_1 \sqrt{2} \left( \beta \sum_{n=0}^{N} \frac{\alpha^n}{k^2 + \alpha^{2n}} \right) ikT_k
\]

\[
q^n_k = -\chi_1 \sqrt{2} \beta \frac{\alpha^n}{k^2 + \alpha^{2n}} ikT_k
\]

\[
(k^2 + \alpha^{2n}) q^n_k = (-\chi_1 \sqrt{2} \beta \alpha^n) ikT_k
\]

\[
\frac{\partial^2}{\partial z^2} + \alpha^{2n} q^n(z) = (-\chi_1 \sqrt{2} \beta \alpha^n) \frac{\partial}{\partial z} T(z)
\]

**FIG. 1.** The real and imaginary parts of the normalized response function \( R(\zeta) = -\frac{nT_0}{n_0e} \phi \) vs the normalized frequency \( \zeta \). The solid lines are the exact kinetic result for a Maxwellian, \( R(\zeta) = 1 + \zeta Z(\zeta) \). The dashed lines are from the three-moment fluid model with \( \Gamma = 3, \mu_1 = 0, \) and \( \chi_1 = 2/\sqrt{\pi} \). The dotted lines are from the four-moment model.
Landau-fluid operators

Nonlocal closure for $q(T)$ uses sum of Lorentzians

Representation of $\text{sign}(k)$ by sum of Lorentzians is used; leads to 2$^{\text{nd}}$ order ODE

$$q(z) = \sum q_n(z)$$

$$\left( A_n \frac{\partial^2}{\partial z^2} + B_n \right) q_n = \frac{\partial}{\partial z} T(z)$$

Using B.C. $q=0$ at $z_{\text{min}}$, $z_{\text{max}}$

Tested in stand-alone IDL code:

1) Spectral exact
2) Spectral with Lorentzians
3) Finite-difference with Lorentzians
The fluid moment approach generates an approximation to the kinetic equation that increases in accuracy as more moments are retained.
In order to verify the BOUT++ GLF results, Korean GLF Team member, Dr SS Kim, developed a gyro-fluid non-local eigen-value solver to solve the same exact set of equations as in BOUT++ framework.
Core Gyrofluid Simulations of Ion Temperature Gradient Turbulence

Eigenvalue Solver guides the GLF model development

![Diagram showing Cyclone case, BOUT++, w/o Landau damping, Ottaviani (3+0), eigenvalue w/ Landau damping, Ottaviani (3+0), FULL (gyro-kinetic), GT3D (gyro-kinetic)]